Matematika problémamegoldó szeminárium, 2019. március 12.

http://www.cs.elte.hu/~kosgeza/oktatas/2019tav-prob/

5.1. (a) Let u and v be two nilpotent elements in a commutative ring (with or without unity). Prove that $u+v$ is also nilpotent. (An element u is called nilpotent if there exists a positive integer n for which $u^n=0$.)

(b) Show an example of a (non-commutative) ring R and nilpotent elements $u,v\in R$ such that $u+v$ is not nilpotent.

5.2. Let $(G,\cdot)$ be a finite group of order n. Show that each element of G is a square if and only if n is odd.

5.3. Given real numbers $0=x_1<x_2<\cdots<x_{2n}<x_{2n+1}=1$ such that $x_{i+1}-x_i\le h$ for $1\le i\le 2n$, show that

$$\frac{1-h}{2}<\sum_{i=1}^{n}x_{2i}(x_{2i+1}-x_{2i-1})<\frac{1+h}{2}.$$

5.4. Suppose that (a_n) is a sequence of real numbers such that the series

$$\sum_{n=1}^{\infty}\frac{a_n}{n}$$

is convergent. Show that the sequence

$$b_n=\frac{\sum_{j=1}^{n}a_j}{n}$$

is convergent and find its limit.

5.5. Two players play the following game: Let n be a fixed integer greater than 1. Starting from number $k=2$, each player has two possible moves: either replace the number k by $k+1$ or by $2k$. The player who is forced to write a number greater than n loses the game. Which player has a winning strategy for which n?

5.6. Let $A=[a_{ij}]_{n\times n}$ be a matrix with nonnegative entries such that

$$\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}=n.$$

(a) Prove that $|\det A|\le 1$.

(b) If $|\det A|=1$ and $\lambda\in\mathbb{C}$ is an arbitrary eigenvalue of A, show that $|\lambda|=1$.

(We call $\lambda\in\mathbb{C}$ an eigenvalue of A if there exists a non-zero vector $x\in\mathbb{C}^n$ such that $Ax=\lambda x$.)

5.7. For a function $f\colon[0,1]\to\mathbb{R}$ the secant of f at points $a,b\in[0,1]$, $a<b$, is the line in $\mathbb{R}^2$ passing through $\big(a,f(a)\big)$ and $\big(b,f(b)\big)$. A function is said to intersect its secant at a,b if there exists a point $c\in(a,b)$ such that $\big(c,f(c)\big)$ lies on the secant of f at a,b.

1. Find the set $\mathcal{F}$ of all continuous functions f such that for any $a,b\in[0,1],a<b$, the function f interescts its secant at a,b.

2. Does there exist a continuous function $f\notin\mathcal{F}$ such that for any rational $a,b\in[0,1],a<b$, the function f intersects its secant at a,b?

5.8. Bizonyítsuk be, hogy minden $c<e$ számhoz léteznek olyan $a_1,a_2,\ldots$ nemnegatív számok, hogy bármely $1=n_0<n_1<n_2<\ldots$ indexekre

$$\sum_{i=0}^{\infty}n_{i+1}a_{n_i}>c\sum_{n=1}^{\infty}a_n.$$

5.9. (Írásban, angolul beadható.) Let $f\colon[0,+\infty)\to\mathbb{R}$ be a strictly convex continuous function such that

$$\lim_{x\to+\infty}\frac{f(x)}{x}=+\infty.$$

Prove that the improper integral $\int_0^{+\infty}\sin\big(f(x)\big)\mathrm{d}x$ is convergent but not absolutely convergent.