Matematika problémamegoldó szeminárium, 2019. április 9.

http://www.cs.elte.hu/~kosgeza/oktatas/2019tav-prob/

9.1. a) Is it true that for every non-empty set A and every associative operation $*: A \times A \to A$ the conditions

$$x * x * y = y \qquad \text{and} \qquad y * x * x = y \qquad \text{for every } x, y \in A$$

imply commutativity of $*$?

b) Is it true that for every non-empty set A and every associative operation $*: A \times A \to A$ the condition

$$x * x * y = y \qquad \text{for every } x, y \in A$$

implies commutativity of $*$?

9.2. Let $\{a_n\}_{n=0}^{\infty}$ be a sequence given recursively by

$$a_0 = 1 \qquad a_{n+1} = \frac{7a_n + \sqrt{45a_n^2 - 36}}{2}, \qquad (n = 0, 1, \ldots).$$

Show that the following statements hold for all positive integers n:

a) a_n is a positive integer.

b) $a_n a_{n+1} - 1$ is the square of an integer.

9.3. Find all twice differentiable functions $f: \mathbb{R} \to \mathbb{R}$ such that

$$f''(x)\cos(f(x)) \geq (f'(x))^2 \sin(f(x))$$

for every $x \in \mathbb{R}$.

9.4. A triplet of polynomials $u, v, w \in \mathbb{R}[x, y, z]$ is called *smart* if there exist polynomials $P, Q, R \in \mathbb{R}[x, y, z]$ such that the following polynomial identity holds:

$$u^{2019}P + v^{2019}Q + w^{2019}R = 2019.$$

a) Is the triplet of polynomials $u = x + 2y + 3$, $v = y + z + 2$, $w = x + y + z$ smart?

b) Is the triplet of polynomials $u = x + 2y + 3$, $v = y + z + 2$, $w = x + y - z$ smart?

9.5. For an invertible n×n matrix M with integer entries we define a sequence $S_M = \{M_i\}_{i=0}^{\infty}$ by the recurrence

$$M_0 = M, \qquad M_{i+1} = (M_i^T)^{-1} M_i, \qquad (i = 0, 1, \ldots).$$

Find the smallest integer $n \geq 2$ for which there exists a normal $n \times n$ matrix M with integer entries such that its sequence S_M is non-constant and has period $P = 7$, i.e., $M_{i+7} = M_i$ for all $i = 0, 1, \ldots$ (M^T means the transpose of a matrix M. A square matrix M is called normal if $M^T M = M M^T$ holds.)

9.6. Let p be an even non-negative continuous function with $\int_{\mathbb{R}} p(x)dx = 1$ and let n be a positive integer. Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent identically distributed random variables with density function p. Define

$$X_0 = 0, \quad X_1 = X_0 + \xi_1, \quad X_2 = X_1 + \xi_2, \quad \ldots \quad X_n = X_n - 1 + \xi_n.$$

Prove that the probability that all the random variables $X_1, X_2, \ldots, X_{n-1}$ lie between X_0 and X_n equals $\frac{1}{n}$.

9.7. Determine the largest constant $K \geq 0$ such that

$$\frac{a^a(b^2 + c^2)}{(a^a - 1)^2} + \frac{b^b(c^2 + a^2)}{(b^b - 1)^2} + \frac{c^c(a^2 + b^2)}{(c^c - 1)^2} \geq K \left(\frac{a + b + c}{abc - 1}\right)^2$$

holds for all positive real numbers a, b, c such that $ab + bc + ca = abc$.

9.8. Let $D = \{z \in \mathbb{C}: \operatorname{Im} z > 0, \operatorname{Re} z > 0\}$. Let $n \geq 1$ and $a_1, \ldots, a_n \in D$ be distinct complex numbers. Define

$$f(z) = z \cdot \prod_{j=1}^{n} \frac{z - a_j}{z - \overline{a_j}}.$$

Prove that f' has at least one root in D.

9.9. (Írásban, angolul beadható.) Suppose $f: \mathbb{R} \to \mathbb{R}$ is a two times differentiable function satisfying $f(0) = 1$, $f'(0) = 0$, and for all $x \in [0, \infty)$

$$f''(x) - 5f'(x) + 6f(x) \geq 0.$$

Prove that for all $x \in [0, \infty)$

$$f(x) \geq 3e^{2x} - 2e^{3x}.$$