SHORTLIST
2018
INTERNATIONAL MONSTERS' OLYMPIAD
The Real
Shortlisted Problems

Problem Selection Committee:

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Problems

Algebra

A1. Determine the maximal possible value of

$$|a - q| \cdot |b - q| \cdot |c - q| \cdots |z - q|$$

if \(a, b, c, \ldots, z\) are real numbers in the interval \([0, 1]\).

A2. Let \(f : \mathbb{N} \rightarrow \mathbb{N}\) be a function such that

$$\left| f(n) - \frac{\sqrt{5} + 1}{2} n \right| < \frac{\sqrt{5} - 1}{2}.$$

Prove that \(f(f(n)) = f(n) + n\).

A3. Find all pairs \(f(x), g(x)\) of polynomials with integer coefficients satisfying

$$f(g(x)) = x^{2007} + 2x + 1.$$

A4. For positive real numbers \(a, b, c\) prove the inequality

$$(a^3b + b^3c + c^3a) + 4(ab^3 + bc^3 + ca^3) \geq 4(a^2b^2 + b^2c^2 + c^2a^2) + abc(a + b + c).$$

A5. Let \(a_1, a_2, \ldots, a_N\) be positive reals. Prove that there exists a sequence \(1 = n_0 < n_1 < \cdots < n_k = N + 1\) of integers such that

$$n_1a_{n_0} + n_2a_{n_1} + \cdots + n_k a_{n_{k-1}} < 3(a_1 + a_2 + \cdots + a_N).$$

A6. Let \(n \geq 2\) and let \(u_1 = 1, u_2, \ldots, u_n\) be arbitrary complex numbers with absolute values at most 1 and let

$$f(x) = (x - u_1)(x - u_2)\cdots(x - u_n).$$

Prove that the polynomial \(f'(x)\) has a root with a non-negative real part.

A7. Prove that

$$\left( \frac{1 + a}{2} \right)^{2x(y+x)} \left( \frac{1 + b}{2} \right)^{2y(x+y)} \geq a^{x^2} b^{y^2} \left( \frac{a + b}{2} \right)^{2xy}$$

holds for all real numbers \(a, b > 0\) and \(x, y\).
Combinatorics

C1. Jack Potter is an addicted gambler. Yesterday, he threw 1000 Romanian Lei in a one-armed bandit machine without his family knowing about it. To make things worse, he took that money from the amount that the family reserved for food. To avoid the affair being revealed, today he is taking the remaining 2000 Lei of the family, too, and goes back to the casino to play roulette. Since he does not want to risk too much, he is betting 50 Lei on red or on black in each game. If he wins, which has a probability of $18/37$, then he gets another 50 Lei. Otherwise he loses his bet. He stops playing either when he has gathered a total of 3000 Lei — in that case he can put all the money back in its place — or when he has lost all the money. What is the probability that Jack Potter will manage to gather the 3000 Lei?

C2. The numbers $1, 2, \ldots, 2018$ are colored with 3 colors such that each color is used at most 1009 times. Let $A$ be set of all (ordered) 4-tuples $(x, y, z, w)$, consisting of such numbers, such that $x + y + z + w \equiv 0 \pmod{2018}$ and $x, y, z, w$ have the same color. Similarly, let $B$ be the set of all (ordered) 4-tuples $(x, y, z, w)$ such that $x + y + z + w \equiv 0 \pmod{2018}$, the numbers $x, y$ have the same color, $z$ and $w$ have the same color, but these two colors are distinct. Prove that $|A| \leq |B|$.

C3. Let $m$ and $n$ be two positive integers such that $\gcd(m + 1, n + 1) = 1$. The cells of an $m \times n$ board are colored in black and white like a chessboard. Each white cell contains a nonnegative integer. The Minesweeper needs to put mines into black cells, any nonnegative number of mines in each, so that the number in every white cell should equal the number of mines in adjacent black cells. Prove that this Minesweeper puzzle has at most one solution.

C4. For every finite, nonempty set $H$ of positive integers, denote by $\gcd(H)$ the greatest common divisor of the elements in $H$. Show that if $A$ is a finite, nonempty set of positive integers then the following inequality holds:

$$\sum_{\emptyset \neq H \subseteq A} (-2)^{|H|-1} \gcd(H) > 0.$$  

C5. It is known that it is possible to draw the complete graph with 7 vertices on the surface of a torus. 7 points are marked on the side of a mug. We want to connect each pair of points with a curve, so that the curves have no interior points in common. What minimum number of these curves need to lead across the handle of the mug?

C6. Does there exist such a lattice rectangle which can be decomposed into lattice pentagons congruent to the one shown in the Figure?
C7. An infinite tape contains the decimal number

\[ 0.1234567891011121314 \ldots, \]

where the decimal point is followed by the decimal representations of all positive integers in increasing order. Determine all positive real \( \alpha \) such that one can erase some digits from the tape so that the following two conditions are satisfied:

(i) The remaining digits form a decimal representation of a rational number;
(ii) For any positive integer \( n \), at most \( \alpha n \) among the first \( n \) digits are removed.

C8. Ginger and Rocky play the following game. First Ginger hides two bones in the corners of a rectangular garden. She may dig 45 cm deep altogether, that is, she may either hide the two bones in two different corners, where the sum of their depths may be at most 45 cm, or she may hide them in the same corner, both bones at a maximum depth of 45 cm. She levels the ground carefully so that it is impossible to see where she has dug. Then Rocky may dig holes with a total depth of 1 m. Rocky’s goal is to maximize the probability of finding both bones, while Ginger’s goal is to maximize the probability of keeping at least one for herself.

(a) Show that if Ginger plays well she can achieve a probability of more than \( \frac{1}{2} \) for at least one bone remaining hidden, independently of Rocky’s search strategy.
(b) What are the chances of Ginger’s success if both dogs play optimally?
Geometry

G1. The bus outlined in the figure is transferring the members of the Problem Selection Committee and their 1024 gigapists from Bucharest to Cluj. In which direction is Cluj? To the left, or to the right? Justify your answer.

G2. Let $\Omega$ be the circumcircle of a triangle $ABC$. Let $A_0$, $B_0$, and $C_0$ be the midpoints of the arcs $BAC$, $CBA$, and $ACB$, respectively. Let $A_1$, $B_1$, and $C_1$ be the Feuerbach points in the triangles $AB_0C_0$, $A_0BC_0$, and $A_0B_0C$, respectively. Prove that the triangles $A_0B_0C_0$ and $A_1B_1C_1$ are similar.

G3. Given a convex quadrilateral $ABCD$ such that $AB = CD$, $\angle ABC = 90^\circ$ and $\angle BCD = 100^\circ$. The perpendicular bisectors of the segments $AD$ and $BC$ meet at $S$. Compute the angle $\angle ASD$.

G4. The circle $\omega_1$ is internally tangent to the circle $\Omega$ which is externally tangent to $\omega_2$. The common external tangents of $\omega_1$ and $\omega_2$ are $u$ and $v$. The line $u$ is tangent to $\omega_1$ and $\omega_2$ at $P$ and $Q$, respectively, and meets $\Omega$ at $A$ and $B$ in such a way that $B$ lies between $P$ and $Q$. Analogously, the line $v$ is tangent to $\omega_1$ and $\omega_2$ at $R$ and $S$, respectively, and meets $\Omega$ at $C$ and $D$ in such a way that $D$ lies between $R$ and $S$ and $\omega_1$ is tangent to that arc $BD$ of $\Omega$ which does not contain $A$ and $C$.

Show that

$$\frac{AB \cdot AD}{AP \cdot AQ} = \frac{CB \cdot CD}{CR \cdot CS}.$$

G5. A sphere $S$ lies within tetrahedron $ABCD$, touching faces $ABD$, $ACD$, and $BCD$, but having no point in common with plane $ABC$. Let $E$ be the point in the interior of the tetrahedron for which $S$ touches planes $ABE$, $ACE$, and $BCE$ as well. Suppose the line $DE$ meets face $ABC$ at $F$, and let $L$ be the point of $S$ nearest to plane $ABC$. Show that segment $FL$ passes through the centre of the inscribed sphere of tetrahedron $ABCE$.

G6. Let $ABC$ be a scalene acute-angled triangle. The tangents to its circumcircle at points $A$ and $B$ meet the opposite sidelines at $A_1$ and $B_1$, respectively. Let $L$ be Lemoine point of the triangle (where the symmedians meet), and let $P$ be a point inside the triangle such that its projections onto the sides form an equilateral triangle. Prove that $LP \perp A_1B_1$.

G7. In the Cartesian plane call those regions as strips which are bounded by two parallel lines. Define the width of a strip as the side length of the inscribed squares, with sides being parallel to the coordinate axes. Prove that if finitely many strips cover the unit square $0 \leq x, y \leq 1$ then the sum of widths of those strips is at least 1.

G8. Let $O$ be an arbitrary point inside a tetrahedron $ABCD$. Prove that

$$[AOC] \cdot [BOD] \leq [AOB] \cdot [COD] + [AOD] \cdot [BOC].$$

(Here $[XYZ]$ stands for the area of triangle $XYZ$.)
Number Theory

N1. What is the number of the parking slot containing the car?

N2. Prove that if $1 \leq k, \ell < n$ are integers then $\frac{n}{k}$ and $\frac{n}{\ell}$ are not coprime.

N3. Prove that $\frac{7^{7^k+1} + 1}{7^{7^k} + 1}$ is a composite number for every nonnegative integer $k$.

N4. Let $a_1, a_2, \ldots$ be a sequence of integers satisfying

$$a_{n+3} = Aa_{n+2} + Ba_{n+1} + Ca_n$$

for all positive integer $n$, where $A$, $B$, and $C$ are some fixed real numbers. May it happen that for every integer $k$ there exists a unique index $n$ satisfying $a_n = k$?

N5. Let $a > b$ and $n$ be arbitrary positive integers. Prove that $n$ divides $\varphi(a^n - b^n)$.

N6. Let $f(k) = 2^k + 1$ for arbitrary positive integer $k$. Is there any positive integer $n$ which divides $f(f(n))$, but does not divide $f(f(f(n)))$?

N7. For any positive integers $x_1, \ldots, x_n > 1$ we denote

$$[x_1, \ldots, x_n] = \frac{1}{\frac{1}{x_1} - \frac{1}{x_2} - \cdots - \frac{1}{x_n}}.$$

Let $a_1, \ldots, a_k, b_1, \ldots, b_n$ be integers greater than 1. Assume that

$$[a_1, \ldots, a_k] + [b_1, \ldots, b_n] > 1.$$

Prove that there exist positive integers $p \leq k$ and $q \leq n$ such that

$$[a_1, \ldots, a_p] + [b_1, \ldots, b_q] = 1.$$

N8. An odd prime number $q$ and nonzero integer numbers $x$ and $y$ satisfy the relation $x^2 = 8y^q + 1$. Prove that $q$ divides $y - 1$. 
Origin of the problems

**A1:** Folklore

**A2:** KöMaL B.3429. (January 2001); G. Kós

**A3:** KöMaL A.429. (May 2007); Katalin Gyarmati

**A4:** I. Bogdanov. An improved version was used in the Olympiad of the 239th School, St.-Petersburg, 2015

**A5:** KöMaL A.394. (February 2006); based on Küschák 2005/1 by András Bíró

**A6:** KöMaL A.430 (May 2007); G. Kós

**A7:** KöMaL A.616. (April 2014); G. Kós

**C1:** KöMaL B.4419. (January 2011); G. Kós

**C2:** KöMaL A.448. (February 2007); G. Kós

**C3:** Kolmogorov Cup, 2013; E. Lakshtanov

**C4:** KöMaL A.492. (November 2009); Péter P. Pach

**C5:** KöMaL B.4938. (February 2018); G. Kós. Dedicated to the memory of Ákos Császár (1924–2017), chairman of the IMO jury in 1982

**C6:** KöMaL A.344. (April 2004); proposed by M. Kristóf. Reported to be a former IMO problem proposal by Belarus

**C7:** Kolmogorov Cup, 2014; I. Bogdanov

**C8:** KöMaL A.646. (May 2015); Endre Csóka

**G1:** Folklore. The idea that 1 therapist is equivalent with 1024 gigapists is due to Lizanka Oravecz, Hungarian autism activist

**G2:** Kolmogorov Cup, 2014, based on a problem by A. Yakubov

**G3:** KöMaL Gv.2938. (October 1994); based on the folklore fake proof for $90^2 = 100^2$

**G4:** KöMaL A.579. (January 2013); G. Kós

**G5:** KöMaL A.723. (April 2018); G. Kós

**G6:** Kolmogorov Cup, 2002; S. Berlov, D. Shiryaev, A. Smirnov

**G7:** KöMaL A.526. (January 2011); based on Keith Ball: The plank problem for symmetric bodies

**G8:** Kolmogorov Cup, 2011; I. Bogdanov

**N1:** Folklore

**N2:** KöMaL N.29. (April 1994); proposed by Gergely Harcos

**N3:** KöMaL A.622. (September 2014)

**N4:** Kolmogorov Cup, 2012, improved; I. Bogdanov, I. Samoylov

**N5:** KöMaL A.415. (December 2006); Balázs Strenner

**N6:** KöMaL A.562. (April 2012), based on RMM 2012/4

**N7:** Kolmogorov Cup, 2013; MathOverflow.net, answered by A. Ustinov

**N8:** Kolmogorov Cup, 2013; A. Polyansky