SHORTLIST

2018

INTERNATIONAL MONSTERS' OLYMPIAD

The Real

Shortlisted Problems

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Problems

Algebra

A1. Determine the maximal possible value of

$$|a-q|\cdot|b-q|\cdot|c-q|\cdots|z-q|$$

if $a, b, c, \ldots, z$ are real numbers in the interval $[0,1]$.

A2. Let $f : \mathbb{N} \to \mathbb{N}$ be a function such that

$$\left|f(n) - \frac{\sqrt{5}+1}{2}n\right| < \frac{\sqrt{5}-1}{2}.$$

Prove that $f(f(n)) = f(n) + n$.

A3. Find all pairs $f(x)$, $g(x)$ of polynomials with integer coefficients satisfying

$$f\big(g(x)\big) = x^{2007} + 2x + 1.$$

A4. For positive real numbers a, b, c prove the inequality

$$(a^3b + b^3c + c^3a) + 4(ab^3 + bc^3 + ca^3) \geqslant 4(a^2b^2 + b^2c^2 + c^2a^2) + abc(a+b+c).$$

A5. Let $a_1, a_2, \ldots, a_N$ be positive reals. Prove that there exists a sequence $1 = n_0 < n_1 < \cdots < n_k = N+1$ of integers such that

$$n_1 a_{n_0} + n_2 a_{n_1} + \cdots + n_k a_{n_{k-1}} < 3(a_1 + a_2 + \cdots + a_N).$$

A6. Let $n \geqslant 2$ and let $u_1 = 1, u_2, \ldots, u_n$ be arbitrary complex numbers with absolute values at most 1 and let

$$f(x) = (x-u_1)(x-u_2)\ldots(x-u_n).$$

Prove that the polynomial $f'(x)$ has a root with a non-negative real part.

A7. Prove that

$$\left(\frac{1+a}{2}\right)^{2x(x+y)}\left(\frac{1+b}{2}\right)^{2y(x+y)} \geqslant a^{x^2}b^{y^2}\left(\frac{a+b}{2}\right)^{2xy}$$

holds for all real numbers $a, b > 0$ and x, y.

Combinatorics

C1. Jack Potter is an addicted gambler. Yesterday, he threw 1000 Romanian Lei in a one-armed bandit machine without his family knowing about it. To make things worse, he took that money from the amount that the family reserved for food. To avoid the affair being revealed, today he is taking the remaining 2000 Lei of the family, too, and goes back to the casino to play roulette. Since he does not want to risk too much, he is betting 50 Lei on red or on black in each game. If he wins, which has a probability of $18/37$, then he gets another 50 Lei. Otherwise he loses his bet. He stops playing either when he has gathered a total of 3000 Lei — in that case he can put all the money back in its place — or when he has lost all the money. What is the probability that Jack Potter will manage to gather the 3000 Lei?

C2. The numbers $1, 2, \dots, 2018$ are colored with 3 colors such that each color is used at most 1009 times. Let A be set of all (ordered) 4-tuples (x, y, z, w), consisting of such numbers, such that $x+y+z+w \equiv 0 \pmod{2018}$ and x, y, z, w have the same color. Similarly, let B be the set of all (ordered) 4-tuples (x, y, z, w) such that $x+y+z+w \equiv 0 \pmod{2018}$, the numbers x, y have the same color, z and w have the same color, but these two colors are distinct. Prove that $|A| \leqslant |B|$.

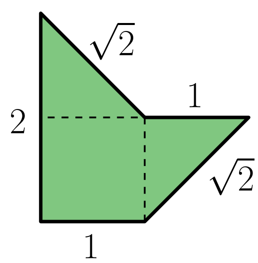
C3. Let m and n be two positive integers such that $\gcd(m+1, n+1) = 1$. The cells of an $m \times n$ board are colored in black and white like a chessboard. Each white cell contains a nonnegative integer. The Minesweeper needs to put mines into black cells, any nonnegative number of mines in each, so that the number in every white cell should equal the number of mines in adjacent black cells. Prove that this Minesweeper puzzle has at most one solution.

C4. For every finite, nonempty set H of positive integers, denote by $\gcd(H)$ the greatest common divisor of the elements in H. Show that if A is a finite, nonempty set of positive integers then the following inequality holds:

$$\sum_{\emptyset \neq H \subseteq A} (-2)^{|H|-1} \gcd(H) > 0.$$

C5. It is known that it is possible to draw the complete graph with 7 vertices on the surface of a torus. 7 points are marked on the side of a mug. We want to connect each pair of points with a curve, so that the curves have no interior points in common. What minimum number of these curves need to lead across the handle of the mug?

C6. Does there exist such a lattice rectangle which can be decomposed into lattice pentagons congruent to the one shown in the Figure?



C7. An infinite tape contains the decimal number

$$0.1234567891011121314\ldots,$$

where the decimal point is followed by the decimal representations of all positive integers in increasing order. Determine all positive real α such that one can erase some digits from the tape so that the following two conditions are satisfied:

(i) The remaining digits form a decimal representation of a rational number;

(ii) For any positive integer n, at most αn among the first n digits are removed.

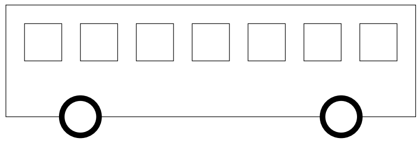
C8. Ginger and Rocky play the following game. First Ginger hides two bones in the corners of a rectangular garden. She may dig 45 cm deep altogether, that is, she may either hide the two bones in two different corners, where the sum of their depths may be at most 45 cm, or she may hide them in the same corner, both bones at a maximum depth of 45 cm. She levels the ground carefully so that it is impossible to see where she has dug. Then Rocky may dig holes with a total depth of 1 m. Rocky's goal is to maximize the probability of finding both bones, while Ginger's goal is to maximize the probability of keeping at least one for herself.

(a) Show that if Ginger plays well she can achieve a probability of more than 1/2 for at least one bone remaining hidden, independently of Rocky's search strategy.

(b) What are the chances of Ginger's success if both dogs play optimally?

Geometry

G1. The bus outlined in the figure is transferring the members of the Problem Selection Committee and their 1024 gigapists from Bucharest to Cluj. In which direction is Cluj? To the left, or to the right? Justify your answer.



G2. Let Ω be the circumcircle of a triangle ABC. Let A_0, B_0, and C_0 be the midpoints of the arcs BAC, CBA, and ACB, respectively. Let A_1, B_1, and C_1 be the Feuerbach points in the triangles AB_0C_0, A_0BC_0, and A_0B_0C, respectively. Prove that the triangles $A_0B_0C_0$ and $A_1B_1C_1$ are similar.

G3. Given a convex quadrilateral $ABCD$ such that $AB = CD$, $\angle ABC = 90°$ and $\angle BCD = 100°$. The perpendicular bisectors of the segments AD and BC meet at S. Compute the angle $\angle ASD$.

G4. The circle ω_1 is internally tangent to the circle Ω which is externally tangent to ω_2. The common external tangents of ω_1 and ω_2 are u and v. The line u is tangent to ω_1 and ω_2 at P and Q, respectively, and meets Ω at A and B in such a way that B lies between P and Q. Analogously, the line v is tangent to ω_1 and ω_2 at R and S, respectively, and meets Ω at C and D in such a way that D lies between R and S and ω_1 is tangent to that arc BD of Ω which does not contain A and C.

Show that
$$\frac{AB \cdot AD}{AP \cdot AQ} = \frac{CB \cdot CD}{CR \cdot CS}.$$

G5. A sphere $\mathcal{S}$ lies within tetrahedron $ABCD$, touching faces ABD, ACD, and BCD, but having no point in common with plane ABC. Let E be the point in the interior of the tetrahedron for which $\mathcal{S}$ touches planes ABE, ACE, and BCE as well. Suppose the line DE meets face ABC at F, and let L be the point of $\mathcal{S}$ nearest to plane ABC. Show that segment FL passes through the centre of the inscribed sphere of tetrahedron $ABCE$.

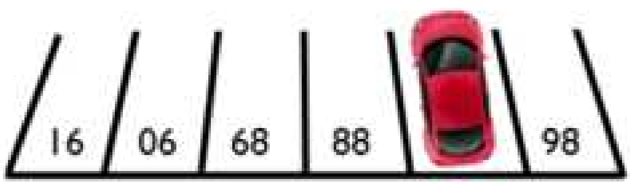
G6. Let ABC be a scalene acute-angled triangle. The tangents to its circumcircle at points A and B meet the opposite sidelines at A_1 and B_1, respectively. Let L be Lemoine point of the triangle (where the symmedians meet), and let P be a point inside the triangle such that its projections onto the sides form an equilateral triangle. Prove that $LP \perp A_1B_1$.

G7. In the Cartesian plane call those regions as *strips* which are bounded by two parallel lines. Define the *width* of a strip as the side length of the inscribed squares, with sides being parallel to the coordinate axes. Prove that if finitely many strips cover the unit square $0 \leqslant x, y \leqslant 1$ then the sum of widths of those strips is at least 1.

G8. Let O be an arbitrary point inside a tetrahedron $ABCD$. Prove that
$$[AOC] \cdot [BOD] \leqslant [AOB] \cdot [COD] + [AOD] \cdot [BOC].$$
(Here $[XYZ]$ stands for the area of triangle XYZ.)

Number Theory

N1. What is the number of the parking slot containing the car?



N2. Prove that if $1 \leqslant k, \ell < n$ are integers then $\binom{n}{k}$ and $\binom{n}{\ell}$ are not coprime.

N3. Prove that $\dfrac{7^{7^{k+1}}+1}{7^{7^k}+1}$ is a composite number for every nonnegative integer k.

N4. Let $a_1, a_2, \ldots$ be a sequence of integers satisfying

$$a_{n+3} = Aa_{n+2} + Ba_{n+1} + Ca_n \qquad \text{for all positive integer } n,$$

where A, B, and C are some fixed real numbers. May it happen that for every integer k there exists a unique index n satisfying $a_n = k$?

N5. Let $a > b$ and n be arbitrary positive integers. Prove that n divides $\varphi(a^n - b^n)$.

N6. Let $f(k) = 2^k + 1$ for arbitrary positive integer k. Is there any positive integer n which divides $f\big(f(n)\big)$, but does not divide $f\big(f\big(f(n)\big)\big)$?

N7. For any positive integers $x_1, \ldots, x_n > 1$ we denote

$$[x_1, \ldots, x_n] = \cfrac{1}{x_1 - \cfrac{1}{x_2 - \cfrac{1}{\ddots - \cfrac{1}{x_n}}}}.$$

Let $a_1, \ldots, a_k, b_1, \ldots, b_n$ be integers greater than 1. Assume that

$$[a_1, \ldots, a_k] + [b_1, \ldots, b_n] > 1.$$

Prove that there exist positive integers $p \leqslant k$ and $q \leqslant n$ such that

$$[a_1, \ldots, a_p] + [b_1, \ldots, b_q] = 1.$$

N8. An odd prime number q and nonzero integer numbers x and y satisfy the relation $x^2 = 8y^q + 1$. Prove that q divides $y - 1$.

Origin of the problems

A1: Folklore
A2: KöMaL B.3429. (January 2001); G. Kós
A3: KöMaL A.429. (May 2007); Katalin Gyarmati
A4: I. Bogdanov. An improved version was used in the Olympiad of the 239th School, St.-Petersburg, 2015
A5: KöMaL A.394. (February 2006); based on Kürschák 2005/1 by András Bíró
A6: KöMaL A.430 (May 2007); G. Kós
A7: KöMaL A.616. (April 2014); G. Kós
C1: KöMaL B.4419. (January 2011); G. Kós
C2: KöMaL A.448. (February 2007); G. Kós
C3: Kolmogorov Cup, 2013; E. Lakshtanov
C4: KöMaL A.492. (November 2009); Péter P. Pach
C5: KöMaL B.4938. (February 2018); G. Kós. Dedicated to the memory of Ákos Császár (1924–2017), chairman of the IMO jury in 1982
C6: KöMaL A.344. (April 2004); proposed by M. Kristóf. Reported to be a former IMO problem proposal by Belarus
C7: Kolmogorov Cup, 2014; I. Bogdanov
C8: KöMaL A.646. (May 2015); Endre Csóka
G1: Folklore. The idea that 1 t(h)erapist is equivalent with 1024 gigapists is due to Lizanka Oravecz, Hungarian autism activist
G2: Kolmogorov Cup, 2014, based on a problem by A. Yakubov
G3: KöMaL Gy.2938. (October 1994); based on the folklore fake proof for $90^\circ = 100^\circ$
G4: KöMaL A.579. (January 2013); G. Kós
G5: KöMaL A.723. (April 2018); G. Kós
G6: Kolmogorov Cup, 2002; S. Berlov, D. Shiryaev, A. Smirnov
G7: KöMaL A.526. (January 2011); based on Keith Ball: The plank problem for symmetric bodies
G8: Kolmogorov Cup, 2011; I. Bogdanov
N1: Folklore
N2: KöMaL N.29. (April 1994); proposed by Gergely Harcos
N3: KöMaL A.622. (September 2014)
N4: Kolmogorov Cup, 2012, improved; I. Bogdanov, L Samoylov
N5: KöMaL A.415. (December 2006); Balázs Strenner
N6: KöMaL A.562. (April 2012), based on RMM 2012/4
N7: Kolmogorov Cup, 2013; MathOverFlow.net, answered by A. Ustinov
N8: Kolmogorov Cup, 2013; A. Polyansky