Matematika problémamegoldó szeminárium, 2016. március 22.

http://www.cs.elte.hu/~kosgeza/oktatas/2016tav-prob/

1. (a) Let u and v be two nilpotent elements in a commutative ring (with or without unity). Prove that $u+v$ is also nilpotent. (An element u is called nilpotent if there exists a positive integer n for which $u^n = 0$.)

(b) Show an example of a (non-commutative) ring R and nilpotent elements $u, v \in R$ such that $u+v$ is not nilpotent.

2. Let $(G, \cdot)$ be a finite group of order n. Show that each element of G is a square if and only if n is odd.

3. Given real numbers $0 = x_1 < x_2 < \cdots < x_{2n} < x_{2n+1} = 1$ such that $x_{i+1} - x_i \leq h$ for $1 \leq i \leq 2n$, show that

$$\frac{1-h}{2} < \sum_{i=1}^{n} x_{2i}(x_{2i+1} - x_{2i-1}) < \frac{1+h}{2}.$$

4. Suppose that (a_n) is a sequence of real numbers such that the series

$$\sum_{n=1}^{\infty} \frac{a_n}{n}$$

is convergent. Show that the sequence

$$b_n = \frac{\sum_{j=1}^{n} a_j}{n}$$

is convergent and find its limit.

5. Two players play the following game: Let n be a fixed integer greater than 1. Starting from number $k = 2$, each player has two possible moves: either replace the number k by $k+1$ or by $2k$. The player who is forced to write a number greater than n loses the game. Which player has a winning strategy for which n?

6. Let $A = [a_{ij}]_{n \times n}$ be a matrix with nonnegative entries such that

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} = n.$$

(a) Prove that $|\det A| \leq 1$.

(b) If $|\det A| = 1$ and $\lambda \in \mathbb{C}$ is an arbitrary eigenvalue of A, show that $|\lambda| = 1$.

(We call $\lambda \in \mathbb{C}$ an eigenvalue of A if there exists a non-zero vector $x \in \mathbb{C}^n$ such that $Ax = \lambda x$.)

7. For a function $f\colon [0,1] \to \mathbb{R}$ the secant of f at points $a, b \in [0,1]$, $a < b$, is the line in $\mathbb{R}^2$ passing through $\big(a, f(a)\big)$ and $\big(b, f(b)\big)$. A function is said to intersect its secant at a, b if there exists a point $c \in (a,b)$ such that $\big(c, f(c)\big)$ lies on the secant of f at a, b.

1. Find the set $\mathcal{F}$ of all continuous functions f such that for any $a, b \in [0,1], a < b$, the function f interescts its secant at a, b.

2. Does there exist a continuous function $f \notin \mathcal{F}$ such that for any rational $a, b \in [0,1], a < b$, the function f intersects its secant at a, b?

8. Bizonyítsuk be, hogy minden $c < e$ számhoz léteznek olyan $a_1, a_2, \ldots$ nemnegatív számok, hogy bármely $1 = n_0 < n_1 < n_2 < \ldots$ indexekre

$$\sum_{i=0}^{\infty} n_{i+1} a_{n_i} > c \sum_{n=1}^{\infty} a_n.$$

9. (Írásban, angolul beadható.) Let $f\colon [0, +\infty) \to \mathbb{R}$ be a strictly convex continuous function such that

$$\lim_{x \to +\infty} \frac{f(x)}{x} = +\infty.$$

Prove that the improper integral $\int_0^{+\infty} \sin\big(f(x)\big) \mathrm{d}x$ is convergent but not absolutely convergent.