Matematika problémamegoldó szeminárium, 2016. május 10.

1. Suppose that f and g are real-valued functions on the real line and $f(r) \leq g(r)$ for every rational r. Does this imply that $f(x) \leq g(x)$ for every real x if

a) f and g are non-decreasing?

b) f and g are continuous?

2. Let ℓ be a line and P a point in $\mathbb{R}^3$. Let S be the set of points X such that the distance from X to ℓ is greater than or equal to two times the distance between X and P. If the distance from P to ℓ is $d > 0$, find the volume of S.

3. Let A, B and C be real square matrices of the same size, and suppose that A is invertible. Prove that if $(A-B)C = BA^{-1}$, then $C(A-B) = A^{-1}B$.

4. In a town every two residents who are not friends have a friend in common, and no one is a friend of everyone else. Let us number the residents from 1 to n and let a_i be the number of friends of the i-th resident. Suppose that $\sum_{i=1}^{n} a_i^2 = n^2 - n$. Let k be the smallest number of residents (at least three) who can be seated at a round table in such a way that any two neighbors are friends. Determine all possible values of k.

5. Let A and B be two complex square matrices such that

$$A^2B + BA^2 = 2ABA.$$

Prove that there exists a positive integer k such that $(AB - BA)^k = 0$.

6. Let p be a prime number and $\mathbb{F}_p$ be the field of residues modulo p. Let W be the smallest set of polynomials with coefficients in $\mathbb{F}_p$ such that

- the polynomials $x+1$ and $x^{p-2} + x^{p-3} + \cdots + x^2 + 2x + 1$ are in W, and
- for any polynomials $h_1(x)$ and $h_2(x)$ in W the polynomial $r(x)$, which is the remainder of $h_1(h_2(x))$ modulo $x^p - x$, is also in W.

How many polynomials are there in W?

7. Let n be a positive integer. An *n-simplex* in $\mathbb{R}^n$ is given by $n+1$ points $P_0, P_1, \ldots, P_n$, called its *vertices*, which do not all belong to the same hyperplane. For every n-simplex S we denote by $v(S)$ the volume of S, and we write $C(S)$ for the center of the unique sphere containing all the vertices of S.

Suppose that P is a point inside an n-simplex S. Let S_i be the n-simplex obtained from S by replacing its i-th vertex by P. Prove that

$$v(S_0)C(S_0) + v(S_1)C(S_1) + \cdots + v(S_n)C(S_n) = v(S)C(S).$$

8. (Írásban, angolul beadható.) Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is a two times differentiable function satisfying $f(0) = 1$, $f'(0) = 0$, and for all $x \in [0, \infty)$

$$f''(x) - 5f'(x) + 6f(x) \geq 0.$$

Prove that for all $x \in [0, \infty)$

$$f(x) \geq 3e^{2x} - 2e^{3x}.$$